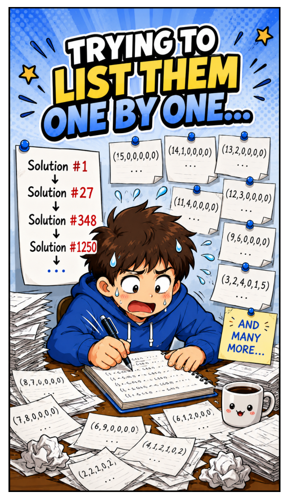
Counting Nonnegative Integer Solutions

Intuition, Stars and Bars, Permutations, Combinations, and Bijections

Visual Motivation

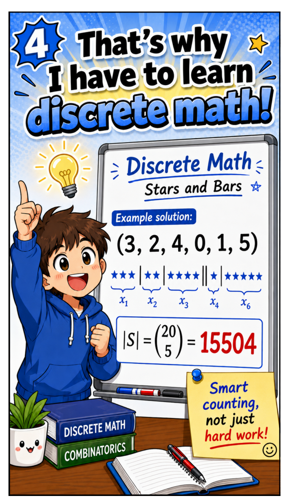
Before doing the mathematics, here is the main idea visually. Trying to list every solution one by one quickly becomes impractical.

(a) Trying to count one by one.



(b) Listing gets messy quickly.

(c) There are too many solutions.



(d) Discrete math gives a systematic method.

Figure 1: Motivation: counting manually is not efficient.

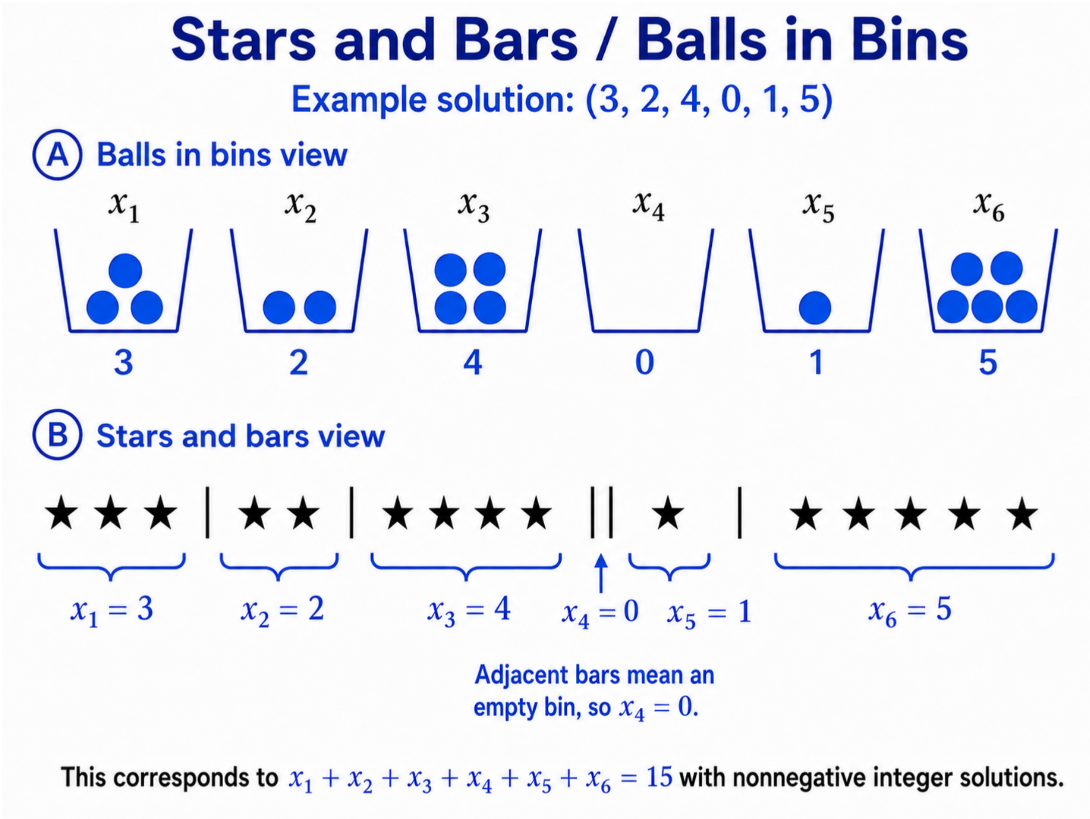


Figure 2: One solution represented as balls in bins and as stars and bars.

Part I

Intuition and Formula

1 Answer First

We want to determine the number of nonnegative integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15, \qquad x_i \geq 0.$$

The answer is

$$\boxed{15504}.$$

This is a **stars and bars** problem.
The general formula is:

$$\text{number of nonnegative integer solutions to } x_1 + x_2 + \cdots + x_k = n$$

is

$$\binom{n+k-1}{k-1}.$$

In our problem,

$$n = 15, \qquad k = 6.$$

Therefore,

$$\binom{15+6-1}{6-1} = \binom{20}{5} = 15504.$$

So the final answer is

$$\boxed{15504}.$$

The rest of this document explains why this formula makes sense.

2 What Are Nonnegative Integer Solutions?

The problem is

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15, \qquad x_i \geq 0.$$

The condition

$$x_i \geq 0$$

means each x_i is a **nonnegative integer**. Therefore, each variable can be

$$0, 1, 2, 3, \ldots$$

So zero is allowed.
This is different from a positive integer condition. If the problem said

$$x_i \geq 1,$$

then every variable would need to be at least 1.
But in this problem, the condition is

$$x_i \geq 0.$$

So we are counting **nonnegative integer solutions**, not positive integer solutions.
For example,

$$(15, 0, 0, 0, 0, 0)$$

is a valid solution because

$$15 + 0 + 0 + 0 + 0 + 0 = 15.$$

Another valid solution is

$$(3, 2, 4, 0, 1, 5),$$

because

$$3 + 2 + 4 + 0 + 1 + 5 = 15.$$

3 Counting as Cardinality

In rigorous mathematics, counting means finding the size of a finite set.

Definition 1 (Cardinality)**.** *Let A be a finite set. The number of elements in A is called the **cardinality** of A, denoted by*

$$|A|.$$

For example, if

$$A = \{a, b, c\},$$

then

$$|A| = 3.$$

In our problem, define the solution set

$$S = \left\{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}_{\geq 0}^6 : x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15\right\}.$$

Here,

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}.$$

So S is the set of all ordered 6-tuples of nonnegative integers whose sum is 15. The goal is to find

$$|S|.$$

That means we want to know how many elements are in S.

4 Basic Counting Principles

Before discussing stars and bars, we first review two basic counting principles.

4.1 Addition Principle

If a task can happen in separate cases, and the cases do not overlap, then we add.

Proposition 1 (Addition Principle)**.** *If A and B are finite sets and*

$$A \cap B = \varnothing,$$

then

$$|A \cup B| = |A| + |B|.$$

For example, suppose a store has 2 red shirts, 4 blue shirts, and 5 black shirts. These categories do not overlap, so the total number of shirts is

$$2 + 4 + 5 = 11.$$

4.2 Multiplication Principle

If a task is done in several steps, and each step has a fixed number of choices, then we multiply.

Proposition 2 (Multiplication Principle)**.** *If A and B are finite sets, then*

$$|A \times B| = |A|\,|B|.$$

Here,

$$A \times B = \{(a, b) : a \in A,\ b \in B\}$$

is the Cartesian product.
For example, suppose there are 3 shirts and 2 pants.
For each shirt, there are 2 choices of pants.
So the number of outfits is

$$3 \cdot 2 = 6.$$

This multiplication principle is the foundation behind permutations and combinations.

5 Permutations

Definition 2 (Permutation)**.** *A **permutation** is an arrangement of objects where the order matters.*

For example,

$$ABC$$

and

$$CBA$$

are different permutations because the order is different.

5.1 Why $n!$?

Suppose we have n distinct objects and want to arrange all of them in a line.
For the first position, there are

$$n$$

choices.
After choosing the first object, there are only

$$n - 1$$

objects remaining for the second position.
After choosing the second object, there are

$$n - 2$$

objects remaining for the third position.
Continuing this way, the number of choices is

$$n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1.$$

This is called $n!$, read as "n factorial."
Thus,

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1.$$

Example 1. *Suppose we have 3 distinct objects:*

$$A, B, C.$$

For the first position, there are 3 choices.
For the second position, there are 2 choices remaining.
For the third position, there is 1 choice remaining.
So the number of arrangements is

$$3\cdot 2\cdot 1 = 6.$$

The six permutations are

$$ABC,\ ACB,\ BAC,\ BCA,\ CAB,\ CBA.$$

5.2 Partial Permutations

Sometimes we do not arrange all n objects. Instead, we choose and arrange only r objects.

Definition 3 (Partial permutation)**.** *The number of ways to choose and arrange r objects from n distinct objects is*

$$P(n,r) = \frac{n!}{(n-r)!}.$$

For example, suppose there are 5 people and 3 chairs.
For the first chair, there are 5 choices.
For the second chair, there are 4 choices.
For the third chair, there are 3 choices.
So the number of ways is

$$5\cdot 4\cdot 3 = 60.$$

Using the formula,

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5\cdot 4\cdot 3 = 60.$$

6 Combinations

Definition 4 (Combination)**.** *A **combination** is a selection of objects where the order does not matter.*

For example,

$$\{A, B\}$$

and

$$\{B, A\}$$

represent the same combination.
The order changed, but the selected group did not change.

Definition 5 (Binomial Coefficient)**.** *The number of ways to choose r objects from n distinct objects is denoted by*

$$\binom{n}{r}$$

and is given by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

6.1 Why Do We Divide by $r!$?

First suppose order matters.
Then choosing and arranging r objects from n objects gives

$$P(n, r) = \frac{n!}{(n-r)!}.$$

But in a combination, order does not matter.
Each selected group of r objects can be arranged in

$$r!$$

different orders.
Therefore, if we first count ordered arrangements, then each unordered group is counted $r!$ times.
So we divide by $r!$:

$$\binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}.$$

Example 2. *Choose 2 objects from*

$$\{A, B, C\}.$$

The possible combinations are

$$\{A, B\}, \quad \{A, C\}, \quad \{B, C\}.$$

So there are 3 combinations.
Using the formula,

$$\binom{3}{2} = \frac{3!}{2!1!} = 3.$$

7 Permutation vs Combination

The key question is:

Does changing the order create a new outcome?

If yes, use permutations.
If no, use combinations.
For permutations,

$$ABC \neq CBA.$$

For combinations,

$$\{A, B, C\} = \{C, B, A\}.$$

In the stars and bars problem, we will eventually choose positions for bars among 20 positions.

Since we only care which positions are chosen, not the order in which we choose them, we use combinations.

8 Balls in Bins

Return to the original equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15, \qquad x_i \geq 0.$$

We can interpret this as:

putting 15 identical balls into 6 distinct bins.

The variables represent the number of balls in each bin:

$$x_1 = \text{number of balls in bin 1},$$

$$x_2 = \text{number of balls in bin 2},$$

and so on.

The balls are identical because we only care about how many balls are in each bin.

The bins are distinct because the variables are labeled:

$$x_1, x_2, x_3, x_4, x_5, x_6.$$

For example, the solution

$$(3, 2, 4, 0, 1, 5)$$

means

$$x_1 = 3, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = 0, \quad x_5 = 1, \quad x_6 = 5.$$

So bin 1 has 3 balls, bin 2 has 2 balls, bin 3 has 4 balls, bin 4 has 0 balls, bin 5 has 1 ball, and bin 6 has 5 balls.

The total is

$$3 + 2 + 4 + 0 + 1 + 5 = 15.$$

Since $x_i \geq 0$, empty bins are allowed.

9 Stars and Bars

The balls in bins problem can be represented using **stars and bars**.

The stars represent balls.

The bars represent dividers between bins.

Since we have 15 balls, we use 15 stars.

Since we have 6 bins, we need 5 bars to separate the bins.

For example,

$$\star\star\star \mid \star\star \mid \star\star\star\star \mid\mid \star \mid \star\star\star\star\star$$

represents

$$(3, 2, 4, 0, 1, 5).$$

The correspondence is:

$$\star\star\star \quad \Rightarrow \quad x_1 = 3,$$

$$\star\star \quad \Rightarrow \quad x_2 = 2,$$

$$\star\star\star\star \quad \Rightarrow \quad x_3 = 4,$$

$$|| \quad \Rightarrow \quad x_4 = 0,$$

$$\star \quad \Rightarrow \quad x_5 = 1,$$

$$\star\star\star\star\star \quad \Rightarrow \quad x_6 = 5.$$

The adjacent bars

$$||$$

mean there is an empty bin between them. That is why

$$x_4 = 0.$$

10 Counting the Stars-and-Bars Strings

Now we count the stars-and-bars strings.

Each string has 15 stars and 5 bars.

So the total number of symbols is

$$15 + 5 = 20.$$

We are arranging a multiset consisting of 15 identical stars and 5 identical bars.

If all 20 symbols were distinct, then there would be

$$20!$$

arrangements.

However, the 15 stars are identical. Rearranging the stars among themselves does not produce a new string, so we divide by

$$15!.$$

Also, the 5 bars are identical. Rearranging the bars among themselves does not produce a new string, so we divide by

$$5!.$$

Therefore, the number of distinct stars-and-bars strings is

$$\frac{20!}{15!5!}.$$

This is the same as

$$\binom{20}{5}.$$

So

$$|S| = \frac{20!}{15!5!} = \binom{20}{5}.$$

Now compute:

$$\binom{20}{5} = \frac{20!}{5!15!} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}.$$

Thus,

$$\binom{20}{5} = 15504.$$

Therefore,

$$\boxed{|S| = 15504}.$$

11 General Stars and Bars Formula

The original problem is a special case of a general theorem.

Theorem 1 (Stars and Bars)**.** *The number of nonnegative integer solutions to*

$$x_1 + x_2 + \cdots + x_k = n$$

is

$$\binom{n+k-1}{k-1}.$$

Proof. Use n stars to represent the total number being distributed.
To split the stars into k groups, we need $k-1$ bars.
So we arrange

$$n + (k-1) = n + k - 1$$

symbols in total.
We choose which $k-1$ positions contain bars.
Therefore, the number of arrangements is

$$\binom{n+k-1}{k-1}.$$

The rigorous reason this also counts the nonnegative integer solutions is that there is a bijection between solutions and stars-and-bars strings. □

For our problem,

$$n = 15, \qquad k = 6.$$

Therefore,

$$\binom{n+k-1}{k-1} = \binom{15+6-1}{6-1} = \binom{20}{5} = 15504.$$

Part II
Rigorous Meaning and Proof

12 Why the Method Works: Functions

To make the stars and bars method rigorous, we need the idea of a function between two sets.

Definition 6 (Function)**.** *Let A and B be sets. A function*

$$f : A \to B$$

assigns to each element $a \in A$ exactly one element $f(a) \in B$.

In simple language, a function is a rule that sends every input to exactly one output.

Example 3. *Let*

$$A = \{1, 2, 3\}, \qquad B = \{a, b, c\}.$$

Define $f : A \to B$ by

$$f(1) = a, \qquad f(2) = b, \qquad f(3) = c.$$

This is a function because every input in A is assigned exactly one output in B.

13 Injective Functions

Definition 7 (Injective)**.** *A function $f : A \to B$ is called **injective**, or one-to-one, if different inputs always give different outputs.*

In symbols, f is injective if

$$f(a_1) = f(a_2) \implies a_1 = a_2.$$

Informally:

different inputs give different outputs.

Example 4 (Injective Function)**.** *Let*

$$A = \{1, 2, 3\}, \qquad B = \{a, b, c, d\}.$$

Define $f : A \to B$ by

$$f(1) = a, \qquad f(2) = b, \qquad f(3) = c.$$

This function is injective because no two different inputs are sent to the same output.

Example 5 (Not Injective)**.** *Let*

$$A = \{1, 2, 3\}, \qquad B = \{a, b, c\}.$$

Define $g : A \to B$ *by*

$$g(1) = a, \qquad g(2) = a, \qquad g(3) = b.$$

This function is not injective because

$$g(1) = g(2) = a,$$

but

$$1 \neq 2.$$

So two different inputs are sent to the same output.

14 Surjective Functions

Definition 8 (Surjective)**.** *A function* $f : A \to B$ *is called* ***surjective****, or onto, if every element of* B *is hit by at least one element of* A*.*

In symbols, f *is surjective if for every* $b \in B$*, there exists* $a \in A$ *such that*

$$f(a) = b.$$

Informally:

every output is reached.

Example 6 (Surjective Function)**.** *Let*

$$A = \{1, 2, 3\}, \qquad B = \{a, b\}.$$

Define $f : A \to B$ *by*

$$f(1) = a, \qquad f(2) = a, \qquad f(3) = b.$$

This function is surjective because every element of B *is hit.*
The output a *is hit by 1 and 2, and the output* b *is hit by 3.*

Example 7 (Not Surjective)**.** *Let*

$$A = \{1, 2, 3\}, \qquad B = \{a, b, c\}.$$

Define $h : A \to B$ *by*

$$h(1) = a, \qquad h(2) = a, \qquad h(3) = b.$$

This function is not surjective because no input maps to c*.*
So $c \in B$ *is missed.*

15 Bijective Functions

Definition 9 (Bijective)**.** *A function $f : A \to B$ is called* ***bijective*** *if it is both injective and surjective.*

So a bijection is a perfect one-to-one matching between two sets.

Example 8 (Bijective Function)**.** *Let*

$$A = \{1, 2, 3\}, \qquad B = \{a, b, c\}.$$

Define $f : A \to B$ by

$$f(1) = a, \qquad f(2) = b, \qquad f(3) = c.$$

This function is bijective because:

- *it is injective: different inputs give different outputs;*
- *it is surjective: every output in B is reached.*

Proposition 3. *If A and B are finite sets and there exists a bijection*

$$f : A \to B,$$

then

$$|A| = |B|.$$

Proof. Since f is injective, no two different elements of A are sent to the same element of B. So A cannot have more elements than B.

Since f is surjective, every element of B is hit by some element of A. So B cannot have more elements than A.

Therefore,

$$|A| = |B|.$$

□

Remark 1. *A bijection is not the same as a permutation.*

A permutation is an arrangement of objects where order matters.

A bijection is a perfect matching between two sets.

In this problem, we use a bijection to prove that counting integer solutions is the same as counting stars-and-bars strings.

16 The Bijection in Stars and Bars

Recall the solution set

$$S = \left\{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6_{\geq 0} : x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15\right\}.$$

Now define another set:

$$T = \{\text{arrangements of 15 stars and 5 bars}\}.$$

We will show that

$$|S| = |T|.$$

To do this, construct a function

$$f : S \to T.$$

Given a solution

$$(x_1, x_2, x_3, x_4, x_5, x_6) \in S,$$

define $f(x_1, x_2, x_3, x_4, x_5, x_6)$ to be the string

$$\underbrace{\star\star\cdots\star}_{x_1} \mid \underbrace{\star\star\cdots\star}_{x_2} \mid \underbrace{\star\star\cdots\star}_{x_3} \mid \underbrace{\star\star\cdots\star}_{x_4} \mid \underbrace{\star\star\cdots\star}_{x_5} \mid \underbrace{\star\star\cdots\star}_{x_6}.$$

For example,

$$f(3, 2, 4, 0, 1, 5) = \star\star\star \mid \star\star \mid \star\star\star\star \mid\mid \star \mid \star\star\star\star\star.$$

16.1 Why This Function Is Injective

Suppose two solutions are different.

Then at least one coordinate is different.

For example,

$$(3, 2, 4, 0, 1, 5)$$

and

$$(3, 2, 3, 1, 1, 5)$$

are different because the third and fourth coordinates are different.

If a coordinate x_i is different, then the number of stars in the i-th group is different.

Therefore, the resulting stars-and-bars string is different.

So different solutions produce different strings.

Hence, f is injective.

16.2 Why This Function Is Surjective

Take any arrangement of 15 stars and 5 bars.

For example,

$$\star\star\star \mid \star\star \mid \star\star\star\star \mid\mid \star \mid \star\star\star\star\star.$$

We can recover a solution by counting stars in each region:

$$x_1 = \text{number of stars before the first bar,}$$

$$x_2 = \text{number of stars between the first and second bars,}$$

and so on.

For the example,

$$\star\star\star \mid \star\star \mid \star\star\star\star \mid\mid \star \mid \star\star\star\star\star$$

we get

$$(3, 2, 4, 0, 1, 5).$$

Because adjacent bars can occur, it is possible to get zero stars in a region.

That gives a coordinate equal to 0.

Therefore, every arrangement of 15 stars and 5 bars comes from some solution in S.

So f is surjective.

16.3 Conclusion

Since f is both injective and surjective, f is bijective.

Therefore,

$$|S| = |T|.$$

So instead of counting solutions directly, we can count arrangements of 15 stars and 5 bars.

17 Final Computation Revisited

Since

$$|S| = |T|,$$

and T is the set of arrangements of 15 stars and 5 bars, we count T.

Each string in T has

$$15 + 5 = 20$$

total positions.
Choose 5 positions for the bars:

$$|T| = \binom{20}{5}.$$

Therefore,

$$|S| = \binom{20}{5} = 15504.$$

So the number of nonnegative integer solutions is

$$\boxed{15504}.$$

18 Summary

We wanted to count the number of nonnegative integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15, \qquad x_i \geq 0.$$

We defined the solution set

$$S = \left\{(x_1, \ldots, x_6) \in \mathbb{Z}_{\geq 0}^6 : x_1 + \cdots + x_6 = 15\right\}.$$

Then we interpreted the problem as putting 15 identical balls into 6 distinct bins.
Using stars and bars:

$$15 \text{ stars} + 5 \text{ bars} = 20 \text{ total symbols}.$$

Choosing the 5 bar positions gives

$$|S| = \binom{20}{5}.$$

Therefore,

$$|S| = \binom{20}{5} = 15504.$$

So the final answer is

$$\boxed{15504}.$$