Actuarial Claim Payment Problem

Adapted from Exercise 9.74

Solution write-up by Giodanno Limin

June 2026

Source and Note

This problem is adapted from Exercise 9.74 in:

Klugman, S. A., Panjer, H. H., and Willmot, G. E. *Loss Models: From Data to Decisions*, 5th ed., John Wiley & Sons, 2019.

The original exercise also includes a compound Poisson approximation part, which is omitted here to focus on expected value and variance.

This solution write-up was prepared with the assistance of ChatGPT and reviewed by the author.

Full Question

A group insurance policy covers 1000 employees. The employees are divided into three groups:

Group	Number of employees	Probability of claim	Mean claim amount
1	500	0.02	500
2	250	0.03	750
3	250	0.04	1000

For each employee, the probability of making a claim depends on their group. If a claim occurs, the claim amount follows an exponential distribution with the mean shown in the table.

Let S be the total claim payment from all 1000 employees.

Find

$$\mathbb{E}[S]$$

and

$$\mathrm{Var}(S).$$

Full Solution

We want to find the expected value and variance of the total claim payment S.

Each employee may or may not make a claim. If the employee does not make a claim, the payment is 0. If the employee makes a claim, the payment is a random claim amount.

For one employee, define

$$Y = \text{claim payment from one employee.}$$

Also define the indicator random variable

$$I = \begin{cases} 1, & \text{if the employee makes a claim,} \\ 0, & \text{if the employee does not make a claim.} \end{cases}$$

Since the probability of a claim is p,

$$\mathbb{P}(I = 1) = p$$

and

$$\mathbb{P}(I = 0) = 1 - p.$$

Therefore,

$$I \sim \text{Bernoulli}(p).$$

Let

$$X = \text{claim amount if a claim occurs.}$$

The problem says that X follows an exponential distribution with mean m. To avoid ambiguity, we use the rate parameterization

$$X \sim \text{Exp}(\lambda),$$

where the probability density function is

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0.$$

For this parameterization,

$$\mathbb{E}[X] = \frac{1}{\lambda}.$$

Since the mean claim amount is m, we have

$$m = \frac{1}{\lambda}.$$

Therefore,

$$\lambda = \frac{1}{m}.$$

Thus,

$$X \sim \text{Exp}\left(\frac{1}{m}\right).$$

Since the employee only receives a claim payment if a claim occurs, we can write

$$Y = IX.$$

This works because if no claim occurs, then $I = 0$, so

$$Y = IX = 0 \cdot X = 0.$$

If a claim occurs, then $I = 1$, so

$$Y = IX = 1 \cdot X = X.$$

Therefore,

$$Y = IX$$

correctly represents the claim payment from one employee.

Expected Value for One Employee

We first compute

$$\mathbb{E}[Y].$$

Since

$$Y = IX,$$

we have

$$\mathbb{E}[Y] = \mathbb{E}[IX].$$

Assuming I and X are independent,

$$\mathbb{E}[IX] = \mathbb{E}[I]\mathbb{E}[X].$$

Now compute $\mathbb{E}[I]$. Since $I \sim \text{Bernoulli}(p)$,

$$\mathbb{E}[I] = 0 \cdot \mathbb{P}(I = 0) + 1 \cdot \mathbb{P}(I = 1).$$

Substitute the probabilities:

$$\mathbb{E}[I] = 0(1 - p) + 1(p).$$

Thus,

$$\mathbb{E}[I] = p.$$

Also,

$$\mathbb{E}[X] = m.$$

Therefore,

$$\mathbb{E}[Y] = \mathbb{E}[I]\mathbb{E}[X].$$

Substitute the values:

$$\mathbb{E}[Y] = pm.$$

So the expected claim payment from one employee is

$$\boxed{\mathbb{E}[Y] = pm}.$$

Expected Value for One Group

Now consider one group with n employees.

Let

$$Y_1, Y_2, \dots, Y_n$$

be the claim payments from the n employees in that group.

The total claim payment from this group is

$$S_j = Y_1 + Y_2 + \cdots + Y_n.$$

Taking expectation,

$$\mathbb{E}[S_j] = \mathbb{E}[Y_1 + Y_2 + \cdots + Y_n].$$

By linearity of expectation,

$$\mathbb{E}[S_j] = \mathbb{E}[Y_1] + \mathbb{E}[Y_2] + \cdots + \mathbb{E}[Y_n].$$

All employees in the same group have the same probability of claim p and the same mean claim amount m, so

$$\mathbb{E}[Y_1] = \mathbb{E}[Y_2] = \cdots = \mathbb{E}[Y_n] = pm.$$

Therefore,

$$\mathbb{E}[S_j] = pm + pm + \cdots + pm.$$

There are n terms, so

$$\mathbb{E}[S_j] = npm.$$

Thus,

$$\boxed{\mathbb{E}[S_j] = npm}.$$

Exact Expected Value of Total Claims

Let S_1, S_2, S_3 be the total claim payments from groups 1, 2, and 3 respectively.

Then

$$S = S_1 + S_2 + S_3.$$

Therefore,

$$\mathbb{E}[S] = \mathbb{E}[S_1 + S_2 + S_3].$$

By linearity of expectation,

$$\mathbb{E}[S] = \mathbb{E}[S_1] + \mathbb{E}[S_2] + \mathbb{E}[S_3].$$

For group 1,

$$n_1 = 500, \qquad p_1 = 0.02, \qquad m_1 = 500.$$

Thus,

$$\mathbb{E}[S_1] = n_1 p_1 m_1.$$

Substitute the values:

$$\mathbb{E}[S_1] = 500(0.02)(500).$$

First,

$$500(0.02) = 10.$$

Then,

$$10(500) = 5000.$$

So,

$$\mathbb{E}[S_1] = 5000.$$

For group 2,

$$n_2 = 250, \qquad p_2 = 0.03, \qquad m_2 = 750.$$

Thus,

$$\mathbb{E}[S_2] = n_2 p_2 m_2.$$

Substitute the values:

$$\mathbb{E}[S_2] = 250(0.03)(750).$$

First,

$$250(0.03) = 7.5.$$

Then,

$$7.5(750) = 5625.$$

So,

$$\mathbb{E}[S_2] = 5625.$$

For group 3,

$$n_3 = 250, \qquad p_3 = 0.04, \qquad m_3 = 1000.$$

Thus,

$$\mathbb{E}[S_3] = n_3 p_3 m_3.$$

Substitute the values:

$$\mathbb{E}[S_3] = 250(0.04)(1000).$$

First,

$$250(0.04) = 10.$$

Then,

$$10(1000) = 10000.$$

So,

$$\mathbb{E}[S_3] = 10000.$$

Therefore,

$$\mathbb{E}[S] = 5000 + 5625 + 10000.$$

Adding these values,

$$\mathbb{E}[S] = 20625.$$

Hence,

$$\boxed{\mathbb{E}[S] = 20625}.$$

Variance for One Employee

Now we compute the variance of the claim payment from one employee.

Recall that

$$Y = IX.$$

We use the formula

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2.$$

We already found that

$$\mathbb{E}[Y] = pm.$$

Therefore,

$$(\mathbb{E}[Y])^2 = (pm)^2.$$

So,

$$(\mathbb{E}[Y])^2 = p^2m^2.$$

Now we need to compute

$$\mathbb{E}[Y^2].$$

Since

$$Y = IX,$$

we square both sides:

$$Y^2 = (IX)^2.$$

Therefore,

$$Y^2 = I^2X^2.$$

Since I is an indicator random variable, I can only be 0 or 1. Therefore,

$$I^2 = I.$$

This is true because if $I = 0$, then

$$I^2 = 0^2 = 0 = I.$$

If $I = 1$, then

$$I^2 = 1^2 = 1 = I.$$

Thus,

$$Y^2 = IX^2.$$

Taking expectation,

$$\mathbb{E}[Y^2] = \mathbb{E}[IX^2].$$

Assuming I and X are independent,

$$\mathbb{E}[IX^2] = \mathbb{E}[I]\mathbb{E}[X^2].$$

Since

$$\mathbb{E}[I] = p,$$

we get

$$\mathbb{E}[Y^2] = p\mathbb{E}[X^2].$$

Now we need $\mathbb{E}[X^2]$.
Since

$$X \sim \text{Exp}\left(\frac{1}{m}\right),$$

we know

$$\mathbb{E}[X] = m$$

and

$$\text{Var}(X) = m^2.$$

Using the identity

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2,$$

we substitute

$$\text{Var}(X) = m^2$$

and

$$\mathbb{E}[X] = m.$$

Thus,

$$m^2 = \mathbb{E}[X^2] - m^2.$$

Add m^2 to both sides:

$$2m^2 = \mathbb{E}[X^2].$$

Therefore,

$$\mathbb{E}[X^2] = 2m^2.$$

So,

$$\mathbb{E}[Y^2] = p\mathbb{E}[X^2].$$

Substitute $\mathbb{E}[X^2] = 2m^2$:

$$\mathbb{E}[Y^2] = p(2m^2).$$

Therefore,

$$\mathbb{E}[Y^2] = 2pm^2.$$

Now return to the variance formula:

$$\text{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 .$$

Substitute the expressions:

$$\text{Var}(Y) = 2pm^2 - p^2m^2.$$

Factor out m^2:

$$\text{Var}(Y) = m^2(2p - p^2).$$

Therefore, for one employee,

$$\boxed{\text{Var}(Y) = m^2(2p - p^2)}.$$

Variance for One Group

Now consider one group with n employees.

The total claim payment from the group is

$$S_j = Y_1 + Y_2 + \cdots + Y_n.$$

Assuming the claim payments from different employees are independent,

$$\text{Var}(S_j) = \text{Var}(Y_1 + Y_2 + \cdots + Y_n).$$

For independent random variables, the variance of a sum is the sum of the variances. Therefore,

$$\text{Var}(S_j) = \text{Var}(Y_1) + \text{Var}(Y_2) + \cdots + \text{Var}(Y_n).$$

Each employee in the same group has the same variance:

$$\text{Var}(Y_i) = m^2(2p - p^2).$$

So,

$$\mathrm{Var}(S_j) = m^2(2p - p^2) + m^2(2p - p^2) + \cdots + m^2(2p - p^2).$$

There are n terms, so

$$\mathrm{Var}(S_j) = nm^2(2p - p^2).$$

Thus,

$$\boxed{\mathrm{Var}(S_j) = nm^2(2p - p^2)}.$$

Exact Variance of Total Claims

Since

$$S = S_1 + S_2 + S_3,$$

and assuming the groups are independent,

$$\mathrm{Var}(S) = \mathrm{Var}(S_1 + S_2 + S_3).$$

For independent groups,

$$\mathrm{Var}(S) = \mathrm{Var}(S_1) + \mathrm{Var}(S_2) + \mathrm{Var}(S_3).$$

Now compute each group separately.

Group 1

For group 1,

$$n_1 = 500, \qquad p_1 = 0.02, \qquad m_1 = 500.$$

Using

$$\mathrm{Var}(S_j) = nm^2(2p - p^2),$$

we get

$$\mathrm{Var}(S_1) = 500(500^2)(2(0.02) - (0.02)^2).$$

Compute each part:

$$500^2 = 250000.$$

$$2(0.02) = 0.04.$$

$$(0.02)^2 = 0.0004.$$

Therefore,

$$2(0.02) - (0.02)^2 = 0.04 - 0.0004.$$

$$0.04 - 0.0004 = 0.0396.$$

So,

$$\text{Var}(S_1) = 500(250000)(0.0396).$$

First,

$$500(250000) = 125000000.$$

Then,

$$125000000(0.0396) = 4950000.$$

Therefore,

$$\text{Var}(S_1) = 4{,}950{,}000.$$

Group 2

For group 2,

$$n_2 = 250, \qquad p_2 = 0.03, \qquad m_2 = 750.$$

Using

$$\text{Var}(S_j) = nm^2(2p - p^2),$$

we get

$$\text{Var}(S_2) = 250(750^2)(2(0.03) - (0.03)^2).$$

Compute each part:

$$750^2 = 562500.$$

$$2(0.03) = 0.06.$$

$$(0.03)^2 = 0.0009.$$

Therefore,

$$2(0.03) - (0.03)^2 = 0.06 - 0.0009.$$

$$0.06 - 0.0009 = 0.0591.$$

So,

$$\mathrm{Var}(S_2) = 250(562500)(0.0591).$$

First,

$$250(562500) = 140625000.$$

Then,

$$140625000(0.0591) = 8310937.5.$$

Therefore,

$$\mathrm{Var}(S_2) = 8{,}310{,}937.5.$$

Group 3

For group 3,

$$n_3 = 250, \qquad p_3 = 0.04, \qquad m_3 = 1000.$$

Using

$$\mathrm{Var}(S_j) = nm^2(2p - p^2),$$

we get

$$\mathrm{Var}(S_3) = 250(1000^2)(2(0.04) - (0.04)^2).$$

Compute each part:

$$1000^2 = 1000000.$$

$$2(0.04) = 0.08.$$

$$(0.04)^2 = 0.0016.$$

Therefore,

$$2(0.04) - (0.04)^2 = 0.08 - 0.0016.$$

$$0.08 - 0.0016 = 0.0784.$$

So,

$$\mathrm{Var}(S_3) = 250(1000000)(0.0784).$$

First,

$$250(1000000) = 250000000.$$

Then,

$$250000000(0.0784) = 19600000.$$

Therefore,

$$\mathrm{Var}(S_3) = 19{,}600{,}000.$$

Total Variance

Now add the variances from the three groups:

$$\mathrm{Var}(S) = 4{,}950{,}000 + 8{,}310{,}937.5 + 19{,}600{,}000.$$

First,

$$4{,}950{,}000 + 8{,}310{,}937.5 = 13{,}260{,}937.5.$$

Then,

$$13{,}260{,}937.5 + 19{,}600{,}000 = 32{,}860{,}937.5.$$

Therefore,

$$\boxed{\mathrm{Var}(S) = 32{,}860{,}937.5}.$$

Final Answer

The expected total claim payment is

$$\boxed{\mathbb{E}[S] = 20625}.$$

The variance of the total claim payment is

$$\boxed{\mathrm{Var}(S) = 32{,}860{,}937.5}.$$